Brain teaser

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1 Introduction

Problem:

A plastic ball can break if it is dropped from a building from a level higher than a certain level. A person wants to determine what is the highest level where the ball can be dropped from without breaking. Given two balls, what is a strategy that minimizes the potential number of trials to determine exactly the level after which the balls break. A ball can be reused after a trial if it didn't break.

Solution:

If x is the height of the building, define $v(x)$ as the solution to our problem:

$$v(x) \equiv \min_{s \in \mathcal{S}} \max_{0 \leq n^* \leq x} \{\text{number of trials to discover } n^* \text{ given s}\}$$

where $\mathcal{S}$ is the strategy space and n^* is the 'breaking point' level.

I want to characterize a strategy to discover the exact breaking point by using only the two balls. A strategy is defined by a series of levels n_1, n_2,... at which trials take place. Let n_m be the first trial at which a ball breaks. After any trial $i < m$ we know that $n^* > \max_{1 \leq j < i} n_j$. Since the goal is to minimize the number of trials to discover the breaking point, it cannot be optimum to make trial $i + 1$ at any level lower than $1 + \max_{1 \leq j < i} n_j$. Therefore any optimum strategy would have $n_{i+1} > \max_{1 \leq j < i} n_j$ for any $i < m$. **That implies $n_{i+1} > n_i$.**

Since the ball didn't break at trial $m - 1$, the breaking point n^* must be higher than n_{m-1}. After the m trial we are left with only one ball. We also know that $n^* \in \{n_{m-1} + 1, ..., n_m\}$. **After the first ball breaks, the only strategy to determine the exact breaking point is to start from level $n_{m-1} + 1$ and keep moving one level up at each trial until the ball breaks or we reached level** $n_m - 1$. If at level $n_m - 1$ the ball didn't break, then $n^* = n_m$ The maximum

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number of trials to determine n^* after trial m is $n_m - 1 - n_{m-1}$. The total trials for this strategy are $m + n_m - n_{m-1} - 1$.

Concluding, an optimum strategy would keep increasing the level at which next trial takes place until the first ball breaks. After that, the next level is $n_{m-1} + 1$ and after each trial we move up one level until either the ball breaks or we reach $n_m - 1$.

In order to find the optimal strategy, note that if the last try was at level n and both balls are intact, then we are faced with the same problem as the beginning, with x replaced by $x - n$. Thus now we only have to find $v(x - n)$.

The function $v(x)$ has a straight forward property: $v(x)$ must be monotonically increasing. As x, the number of levels, increases it must be the case that the number of trials cannot not decrease.

Recursive formulation:

The first trial has only two possible outcomes:

1. the ball breaks implying that $m = 1$ and $n^* \in \{1, ..., n_1\}$; therefore

$$\max_{1 \leq n^* \leq n_1} \{\text{number of trials to discover } n^* \text{ given } s^*\} = n_1$$

2. the ball doesn't break and $n^* \in \{n_1 + 1, ..., x\}$; therefore

$$\max_{n_1+1 \leq n^* \leq x} \{\text{number of trials to discover } n^* \text{ given } s^*\} = 1 + v(x - n_1)$$

From the above relations we have

$$v(x) = \max\{n_1, 1 + v(x - n_1)\}$$

where n_1 is the first trial level for any optimal strategy. If $n_1 < 1+v(x-n_1)$ then $v(x) = 1+v(x-n_1)$. In this case we can increase n_1, which leads to a decrease in $v(x - n_1)$, which leads to a decrease in $v(x)$. Therefore in any optimal strategy which minimizes $v(x)$ it cannot be that $n_1 < 1 + v(x - n_1)$.

If $n_1 > 1+v(x-n_1)$ then $v(x) = n_1$. In this case we can decrease n_1, which leads to a decrease in $v(x)$. Therefore in any optimal strategy which minimizes $v(x)$ it cannot be that $n_1 > 1 + v(x - n_1)$.

We established that $v(x) = n_1 = 1 + v(x - n_1)$. Solving recursively, $v(x) = n_1 = 1 + n_2 = 2 + n_3 = ... = k + n_k = k + v(x - n_1 - n_2 - n_3 - ...n_k)$. Where n_i is the number of floors you move up if the first ball does not break after trial $i - 1$. Also n_i is the number of floors you move up at the first step when the the total number of floors is $x - n_1 - ... - n_{i-1}$. Which means $n_i = 1 + n_{i+1}$. Since $v(1) = 1$ then we keep going recursively until $n_k = 1$, which implies that k solves $1 = x - k - (k - 1) - ... - 1$ and $v(x) = k + 1$.

Therefore $v(x)$ solves $v(x) * (v(x) - 1) = 2(x + 1)$, rounded up to the nearest integer. The optimal strategy is: if the last trial was at floor f and there are 2 balls left, then drop a ball from floor $f + v(x - f)$. If it doesn't break go back to previous step, if it breaks then start from floor $f + 1$ and go up one by one.